

# Classification of Cosmic Scale Factor via Noether Gauge Symmetries

Adil Jhangeer <sup>(1)\*</sup>, M. Farasat Shamir <sup>(2)†</sup>,  
Tayyaba Naz <sup>(2) ‡</sup> and Nazish Iftikhar <sup>(2) §</sup>

<sup>(1)</sup>Deanship of Educational Services, Preparatory Year Unit,  
Qassim University,  
P.O. Box 6595, Al- Qassim, Buraidah: 51452  
Kingdom of Saudi Arabia.

<sup>(2)</sup>Department of Sciences and Humanities,  
National University of Computer and Emerging Sciences,  
Lahore Campus, Pakistan.

## Abstract

In this paper, a complete classification of Friedmann-Robertson-Walker (FRW) spacetime by using approximate Noether approach is presented. Considered spacetime is discussed for three different types of universe i.e. flat, open and closed. Different forms of cosmic scale factor  $a$  with respect to the nature of the universe, which posses the nontrivial Noether gauge symmetries (NGS) are reported. The perturbed Lagrangian corresponding to FRW metric in the Noether equation is used to get Noether operators. For different types of universe minimal and maximal set of Noether operators are reported. A list of Noether operators are also computed which is not only independent

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\*adil.jhangeer@gmail.com

†farasat.shamir@nu.edu.pk

‡tayyaba.naz42@yahoo.com

§nazish.iftikhar289@gmail.com

from the choice of the cosmic scale factor but also the choice type of universe. Further, corresponding energy type first integral of motions are also calculated.

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## 1 Introduction

FRW models of the universe are known to be spatially homogeneous and isotropic in nature. This indicates that these models are the best representation of the large scale structure of our present universe. The FRW spacetime is given by

$$ds^2 = dt^2 - a^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right], \quad (1)$$

where  $d\Omega^2 = (d\theta^2 + \sin^2 \theta d\phi^2)$  and the curvature parameter  $k$  is 0, 1 or  $-1$ , which represents flat, open or closed universe respectively. Here  $a$  is a function of cosmic time  $t$  and commonly known as scale factor of universe. It is also called Robertson-Walker scale factor parameter of the Friedmann equations which represents the expansion of the universe. The importance of scale factor can be well understood from the facts that it is used in defining some important terms like the Hubble parameter and red shift. It has been established [1]-[4] that the universe is now expanding with an accelerated rate which means that the second derivative of the scale factor with respect to cosmic time should be positive.

Cosmological solutions in multidimensional model with multiple exponential potential are obtained by Ivashchuk et al. [5]. They classified the solutions with power law and exponential behavior of scale factors. Similar classification of scale factor has been used by Sharif et al. [6] to investigate Bianchi Type  $I$  and  $V$  solutions in  $f(R)$  theory of gravity. They used the ansatz method to find the solutions and classification of scale factor. It would be interesting to classify the scale factor using Noether approach.

A first systematic approach for finding the conserved quantities [7, 8] of the variational problems is given by a German mathematician Emmy

Noether. She reported a relation between the symmetries and conservation laws [8]. These symmetries are called Noether symmetries. She concluded that each symmetry corresponds to a conserved quantity. The idea of conservation laws in mathematics came from devising the physical laws for conserved quantities like mass, energy and momentum.

Conservation laws play a vital role in the field of differential equations. They help to compute the unknown exponent in the similarity solution which cannot be obtained from homogeneous boundary conditions [9]. The conserved quantities are also helpful to control numerical errors in the numerical integration of partial differential equations (PDEs). It would be interesting to mention here that many physical solutions of general relativity field equations do possess some symmetry [10]. Bokhari et al. [11] investigated the symmetries generated by Killing vectors for some spacetimes. They established that Noether symmetries of these spacetimes are more than the symmetries generated by Killing vectors. Thus it was conjectured that the Noether symmetries always provided a larger set of Lie algebra and Killing symmetries formed a subalgebra of Noether symmetries.

Modified theories of gravity have attracted much attention of the researchers during the last decade. Jamil et al. investigated some new exact solutions of static wormholes in the context of  $f(T)$  gravity [12]. The same authors discussed the energy conditions in generalized teleparallel gravity models [13]. In addition to this, violation of first law of thermodynamics is also discussed in  $f(R, T)$  gravity [14]. The resolution of dark matter problem has been studied in  $f(T)$  gravity [15]. Bamba et al. [16] have investigated the generalized second law of thermodynamics with entropy corrections in the field of  $f(T)$  gravity. Similarly many authors explored modified theories in different contexts [17]-[19]. Noether symmetries have also been used to deal many issues in modified theories of gravity. Kucukakca and Camci [20] have obtained the function  $f(R)$  and the scale factor in Palatini  $f(R)$  theory via Noether symmetry approach. Jamil et al. [21] discussed the  $f(R)$ -Tachyon model in the metric formalism and also resulted a zero gauge function. Bianchi type  $I$  cosmology in generalized Saez-Ballester theory has also been studied by using the definition of NGS [22]. Hussain et al. [23] found Noether symmetries for the flat FRW model using the gauge term in metric  $f(R)$  gravity.

Noether symmetries are helpful in recovering some lost conservation laws and symmetry generators of spacetimes [24]. Sharif et al. [25] explored the energy contents of colliding plane waves using approximate Noether sym-

metry approach. It has been concluded that for plane electromagnetic and gravitational waves there does not exist any non-trivial first order symmetry generator. The same authors [26] used approximate symmetries to investigate the energy contents of Bardeen model and stringy charged black hole solutions. Capozziello et al. [27] investigated  $f(R)$  gravity for spherically symmetric spacetime using Noether symmetry. In [28], a new class of Noether symmetries has been reported for spherical symmetry in  $f(R)$  gravity. Paliathanasis et al. [29] reviewed the modified  $f(R)$  gravity models using the Lie and Noether point operators. In a recent paper, Shamir et al. [30] investigated  $f(R)$  gravity using Noether gauge symmetries. For this purpose, Noether symmetry generators for FRW universe and spherically symmetric spacetimes are evaluated. The corresponding conserved quantities are also obtained along with importance and stability criteria of some specific  $f(R)$  models. Here our attention is to discuss the cosmic scale factor  $a$  using NGS approach.

The paper is organized as follows: Section 2 is devoted for the fundamental operators, while in Section 3 classification of scale factor by Noether approach for different types of universe is reported.

## 2 Fundamental operators

Let

$$ds^2 = g_{ij}dx^i dx^j, \quad (2)$$

be a line element then the vector field  $X$  for (2) will be

$$X = \xi(s, x^i) \frac{\partial}{\partial s} + \eta^j(s, x^i) \frac{\partial}{\partial x^j}. \quad (3)$$

The Lagrangian  $L$  for metric (2) is given by [31, 32]

$$L = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j, \quad (4)$$

where dot represent derivative with respect to  $s$ . The Noether equation is:

$$X^{[1]}(L) + LD_s(\xi) = D_s(A), \quad (5)$$

where  $A$  is a gauge function and  $X^{[1]}$  is the first order prolongation and  $D_s$  is called total derivative defined as

$$D_s = \frac{\partial}{\partial s} + \dot{x}^i \frac{\partial}{\partial x^i} . \quad (6)$$

The energy type first integral of motion also known as conserved quantity corresponding to Noether operator  $X$  is defined as

$$I = \xi L + (\eta^i - \xi \dot{x}^i) \frac{\partial L}{\partial \dot{x}^i} - A. \quad (7)$$

### 3 Energy contents for FRW metric

The Lagrangian for FRW spacetimes (1) is

$$L = \dot{t}^2 - \left( \frac{a^2 \dot{r}^2}{1 - kr^2} \right) - a^2 r^2 \dot{\theta}^2 - a^2 r^2 \dot{\phi}^2 \sin^2 \theta. \quad (8)$$

The corresponding vector field  $X$  is defined as

$$X = \xi(s, t, r, \theta, \phi) \frac{\partial}{\partial s} + \eta^1(s, t, r, \theta, \phi) \frac{\partial}{\partial t} + \eta^2(s, t, r, \theta, \phi) \frac{\partial}{\partial r} + \eta^3(s, t, r, \theta, \phi) \frac{\partial}{\partial \theta} + \eta^4(s, t, r, \theta, \phi) \frac{\partial}{\partial \phi}.$$

Next we will discuss three different cases with respect to the value of  $k$ .

#### 3.1 For flat universe ( $k = 0$ )

The Lagrangian (8) for this case becomes

$$L = \dot{t}^2 - a^2 \dot{r}^2 - a^2 r^2 \dot{\theta}^2 - a^2 r^2 \dot{\phi}^2 \sin^2 \theta. \quad (9)$$

Substituting Eq. (9) in Eq. (5) and after some manipulation we get an over-determined system of linear PDEs

$$(i) \ \xi_\phi = 0, \quad (ii) \ \xi_\theta = 0, \quad (iii) \ \xi_r = 0, \quad (iv) \ \xi_t = 0, \quad (v) \ A_s = 0, \quad (10)$$

$$(i) \ 2\eta_s^1 = A_t, \quad (ii) \ -2a^2\eta_s^2 = A_r, \quad (iii) \ -2a^2r^2\eta_s^3 = A_\theta, \quad (11)$$

$$(i) -2a^2r^2 \sin^2 \theta \eta_s^4 = A_\phi, \quad (ii) \eta_r^1 - a^2 \eta_t^2 = 0, \quad (12)$$

$$(i) \eta_\theta^1 - a^2 r^2 \eta_t^3 = 0, \quad (ii) \eta_\phi^1 - a^2 r^2 \sin^2 \theta \eta_t^4 = 0, \quad (13)$$

$$(i) a^2 r^2 \eta_r^3 + a^2 \eta_\theta^2 = 0, \quad (ii) a^2 \eta_\phi^2 + a^2 r^2 \sin^2 \theta \eta_r^4 = 0, \quad (14)$$

$$(i) a^2 r^2 \eta_\phi^3 + a^2 r^2 \sin^2 \theta \eta_\theta^4 = 0, \quad (ii) 2\eta_t^1 - \xi_s = 0, \quad (15)$$

$$a^2 \xi_s - 2a^2 \eta_r^2 - 2aa_t \eta^1 = 0, \quad (16)$$

$$-2a^2 r \eta^2 + a^2 r^2 \xi_s - 2a^2 r^2 \eta_\theta^3 - 2aa_t r^2 \eta^1 = 0, \quad (17)$$

$$a^2 r^2 \sin^2 \theta \xi_s - 2a^2 r^2 \sin^2 \theta \eta_\phi^4 - 2a^2 r \eta^2 \sin^2 \theta - 2a^2 r^2 \sin \theta \cos \theta \eta^3 - 2aa_t r^2 \sin^2 \theta \eta^1 = 0. \quad (18)$$

### 3.1.1 Minimal set of Noether operators

If  $a(t)$  is arbitrary then one can get minimal number of Noether symmetries:

$$\mathbf{X}_{01} = \frac{\partial}{\partial \mathbf{s}}, \quad \mathbf{X}_{02} = \frac{\partial}{\partial \phi}$$

where the gauge function  $A$  is constant. The corresponding energy type first integral of motions associated with  $\mathbf{X}_{01}$  and  $\mathbf{X}_{02}$  are:

$$\mathbf{I}_{01} = -\dot{\mathbf{t}}^2 + \mathbf{a}^2 \dot{\mathbf{r}}^2 + \mathbf{a}^2 \mathbf{r}^2 \dot{\theta} + \mathbf{a}^2 \mathbf{r}^2 \sin^2 \theta \dot{\phi}^2, \quad \mathbf{I}_{02} = -2\mathbf{a}^2 \mathbf{r}^2 \sin^2 \theta \dot{\phi}.$$

### 3.1.2 Five Noether operators

Some different forms of  $a(t)$  which posses five Noether operators are listed in Table: 1 and the set of Noether symmetries consist of  $\mathbf{X}_{01}$ ,  $\mathbf{X}_{02}$

$$\mathbf{X}_{03} = -\cos \theta \frac{\partial}{\partial \mathbf{r}} + \frac{\sin \theta}{\mathbf{r}} \frac{\partial}{\partial \theta}$$

$$\mathbf{X}_{04} = -\cos \phi \sin \theta \frac{\partial}{\partial \mathbf{r}} - \frac{\cos \phi \cos \theta}{\mathbf{r}} \frac{\partial}{\partial \theta} + \frac{\sin \phi}{\mathbf{r} \sin \theta} \frac{\partial}{\partial \phi}$$

Table: 1

No.	$a(t)$
1	$\sin t, \cos t, \tan t$
2	$\sinh t, \cosh t, \tanh t$
3	$\sqrt{\sin t}, \sqrt{\cos t}, \sqrt{\tan t}$
4	$\sqrt{\sinh t}, \sqrt{\cosh t}, \sqrt{\tanh t}$
5	$\sqrt{\ln t}$
6	$\sqrt{ct + b}$
7	$t + \frac{1}{t}$
8	$c^2 b^2 \cosh(\frac{t}{b})^2$

and

$$\mathbf{X}_{05} = \sin \phi \sin \theta \frac{\partial}{\partial \mathbf{r}} + \frac{\sin \phi \cos \theta}{\mathbf{r}} \frac{\partial}{\partial \theta} + \frac{\cos \phi}{\mathbf{r} \sin \theta} \frac{\partial}{\partial \phi}.$$

The conserved quantities corresponding to  $\mathbf{X}_{03}$ ,  $\mathbf{X}_{04}$  and  $\mathbf{X}_{05}$  are:

$$\mathbf{I}_{03} = 2\mathbf{a}^2 \cos \theta \dot{\mathbf{r}} - 2\mathbf{a}^2 \mathbf{r} \sin \theta \dot{\theta}$$

$$\mathbf{I}_{04} = 2\mathbf{a}^2 \cos \phi \sin \theta \dot{\mathbf{r}} + 2\mathbf{a}^2 \mathbf{r} \cos \phi \cos \theta \dot{\theta} - 2\mathbf{a}^2 \mathbf{r} \sin \phi \sin \theta \dot{\phi},$$

$$\mathbf{I}_{05} = -2\mathbf{a}^2 \sin \phi \sin \theta \dot{\mathbf{r}} - 2\mathbf{a}^2 \mathbf{r} \sin \phi \cos \theta \dot{\theta} - 2\mathbf{a}^2 \mathbf{r} \cos \phi \sin \theta \dot{\phi}.$$

### 3.1.3 Six Noether operators

In this case, we will report some of the forms of cosmic scale factors which posses six Noether operators.

(1):  $a(t) = t^n$

For  $n = 2, 4, 6, \dots$  we have  $\mathbf{X}_{01} - \mathbf{X}_{05}$  and

$$X_{06} = s \frac{\partial}{\partial s} + \frac{t}{2} \frac{\partial}{\partial t} - \frac{1}{2} (n-1) r \frac{\partial}{\partial r}$$

The associated energy type first integral of motion will be

$$I_{06} = -st^2 + sa^2 \dot{r}^2 + sa^2 r^2 \dot{\theta}^2 + sa^2 r^2 \sin^2 \theta \dot{\phi}^2 + tt + (n-1)a^2 r \dot{r}.$$

For  $n = 3, 5, 7, 9, \dots$  and  $n = 3 + 2p$ , we have  $\mathbf{X}_{01} - \mathbf{X}_{05}$  and

$$X_{06} = s \frac{\partial}{\partial s} + \frac{t}{2} \frac{\partial}{\partial t} - (p+1)r \frac{\partial}{\partial r}.$$

The corresponding energy type first integral of motion associated with  $X_{06}$  will be

$$I_{06} = -st^2 + sa^2\dot{r}^2 + sa^2r^2\dot{\theta}^2 + sa^2r^2 \sin^2 \theta \dot{\phi}^2 + t\dot{t} + 2(p+1)a^2r\dot{r}.$$

(2):  $a(t) = \sqrt{t^n}$

For  $n = 1, 3, 5, 7, \dots$  we have  $\mathbf{X}_{01} - \mathbf{X}_{05}$  and

$$X_{06} = s \frac{\partial}{\partial s} + \frac{t}{2} \frac{\partial}{\partial t} - \frac{1}{4}(n-2)r \frac{\partial}{\partial r}$$

The associated conserved quantity will be

$$I_{06} = -st^2 + sa^2\dot{r}^2 + sa^2r^2\dot{\theta}^2 + sa^2r^2 \sin^2 \theta \dot{\phi}^2 + t\dot{t} + \frac{1}{2}(n-2)a^2r\dot{r}$$

For  $n = 4, 8, 12, 16, \dots$  we take  $n = 4 + 4p$  and  $p = 0, 1, 2, 3, \dots$

The Noether operators are:

$\mathbf{X}_{01} - \mathbf{X}_{05}$  and

$$X_{06} = s \frac{\partial}{\partial s} + \frac{t}{2} \frac{\partial}{\partial t} - \frac{1}{2}(2p+1)r \frac{\partial}{\partial r}$$

The corresponding conserved quantity will be

$$I_{06} = -st^2 + sa^2\dot{r}^2 + sa^2r^2\dot{\theta}^2 + sa^2r^2 \sin^2 \theta \dot{\phi}^2 + t\dot{t} + (2p+1)a^2r\dot{r}$$

For  $n = 6, 10, 14, 18, 20, \dots$  we take  $n = 6 + 4p$  and  $p = 0, 1, 2, 3, \dots$

The Noether operators are:  $\mathbf{X}_{01} - \mathbf{X}_{05}$  and

$$X_{06} = s \frac{\partial}{\partial s} + \frac{t}{2} \frac{\partial}{\partial t} - (p+1)r \frac{\partial}{\partial r}$$

The associated energy type first integral of motion will be

$$I_{06} = -st^2 + sa^2\dot{r}^2 + sa^2r^2\dot{\theta}^2 + sa^2r^2 \sin^2 \theta \dot{\phi}^2 + t\dot{t} + (p+1)a^2r\dot{r}$$



### 3.1.4 Seven Noether operators

In this case, we will report some of the forms of cosmic scale factors which posses seven Noether operators.

(1):  $a(t)=\alpha t + \beta$

where  $\alpha$  and  $\beta$  are arbitrary constant.

For this case, we have  $\mathbf{X}_{01} - \mathbf{X}_{05}$ , the others two are:

$$X_{06} = \frac{s^2}{2} \frac{\partial}{\partial s} + \frac{s}{2} \left( t + \frac{\beta}{\alpha} \right) \frac{\partial}{\partial t}, \quad (19)$$

$$X_{07} = s \frac{\partial}{\partial s} + \frac{1}{2} \left( t + \frac{\beta}{\alpha} \right) \frac{\partial}{\partial t}, \quad (20)$$

$$A = \frac{1}{2\alpha} \left[ (c_1 t^2 + 2c_4) \alpha + 2c_1 \beta t \right].$$

The associated integral of motions corresponding to  $X_{06}$  and  $X_{07}$  are:

$$I_{06} = -\frac{s^2}{2} \dot{t}^2 + \frac{s^2}{2} a^2 \dot{r}^2 + \frac{s^2}{2} a^2 r^4 \dot{\theta}^2 + \frac{s^2}{2} a^2 r^4 \dot{\phi}^2 \sin^2 \theta + st\dot{t} + \frac{s\beta\dot{t}}{\alpha} + \frac{t^2}{2} + \frac{\beta}{\alpha},$$

$$I_{07} = -s\dot{t}^2 + sa^2\dot{r}^2 + sa^2r^4\dot{\theta}^2 + sa^2r^4\dot{\phi}^2\sin^2\theta + t\dot{t} + \frac{\beta\dot{t}}{\alpha}.$$

(2):  $a(t) = \sqrt{t^2}$

For this case, we have  $\mathbf{X}_{01} - \mathbf{X}_{05}$ ,

$$X_{06} = s \frac{\partial}{\partial s} + \frac{t}{2} \frac{\partial}{\partial t}, \quad X_{07} = \frac{s^2}{2} \frac{\partial}{\partial s} + \frac{st}{2} \frac{\partial}{\partial t}, \quad (21)$$

$$A = \frac{c_1 t^2}{2} + c_4.$$

The associated conserved quantities will be

$$I_{06} = -s\dot{t}^2 + sa^2\dot{r}^2 + sa^2r^2\dot{\theta}^2 + sa^2r^2\sin^2\theta\dot{\phi}^2 + t\dot{t},$$

$$I_{07} = -s^2\dot{t}^2 + s^2a^2\dot{r}^2 + s^2a^2r^2\dot{\theta}^2 + s^2a^2r^2\sin^2\theta\dot{\phi}^2 + 2st\dot{t} - t^2.$$

### 3.1.5 Maximal set of Noether operators

One can obtain the following set of maximal Noether operators for  $a(t) = \text{constant}$  i.e.  $\mathbf{X}_{01} - \mathbf{X}_{03}$

$$X_{04} = \frac{\partial}{\partial t}, \quad X_{05} = \frac{s}{2} \frac{\partial}{\partial t}, \quad (22)$$

$$X_{06} = \left(-\frac{s \cos \theta}{2}\right) \frac{\partial}{\partial r} + \left(\frac{s \sin \theta}{2r}\right) \frac{\partial}{\partial \theta}, \quad X_{07} = r \cos \theta \frac{\partial}{\partial t} + t \cos \theta \frac{\partial}{\partial r} - \left(\frac{t \sin \theta}{r}\right) \frac{\partial}{\partial \theta},$$

$$X_{08} = s \frac{\partial}{\partial s} + \frac{t}{2} \frac{\partial}{\partial t} + \frac{r}{2} \frac{\partial}{\partial r}, \quad X_{09} = \frac{s^2}{2} \frac{\partial}{\partial s} + \frac{st}{2} \frac{\partial}{\partial t} + \frac{sr}{2} \frac{\partial}{\partial r},$$

$$X_{10} = -\left(\frac{s \sin \theta \sin \phi}{2}\right) \frac{\partial}{\partial r} - \left(\frac{s \cos \theta \sin \phi}{2r}\right) \frac{\partial}{\partial \theta} - \left(\frac{s \cos \phi}{2r \sin \theta}\right) \frac{\partial}{\partial \phi}$$

$$X_{11} = -\left(\frac{s \sin \theta \cos \phi}{2}\right) \frac{\partial}{\partial r} - \left(\frac{s \cos \theta \cos \phi}{2r}\right) \frac{\partial}{\partial \theta} + \left(\frac{s \sin \phi}{2r \sin \theta}\right) \frac{\partial}{\partial \phi},$$

$$X_{12} = r \sin \theta \sin \phi \frac{\partial}{\partial t} + t \sin \phi \sin \theta \frac{\partial}{\partial r} + \frac{t \cos \theta \sin \phi}{r} \frac{\partial}{\partial \theta} + \left(\frac{t \cos \phi}{r \sin \theta}\right) \frac{\partial}{\partial \phi},$$

$$X_{13} = r \sin \theta \cos \phi \frac{\partial}{\partial t} + t \cos \phi \sin \theta \frac{\partial}{\partial r} + \frac{t \cos \theta \cos \phi}{r} \frac{\partial}{\partial \theta} - \left(\frac{t \sin \phi}{r \sin \theta}\right) \frac{\partial}{\partial \phi}.$$

The corresponding energy type first integral of motions associated with  $X_{04} - X_{13}$  are

$$I_{04} = 2\dot{t}, \quad I_{05} = s\dot{t} - t,$$

$$I_{06} = \frac{sa^2 \cos \theta \dot{r}}{2} - sa^2 r^2 \sin \theta \dot{\theta} - r \cos \theta,$$

$$\begin{aligned}
I_{07} &= 2r \cos \theta \dot{t} - 2a^2 t \cos \theta \dot{r} - 2a^2 r t \sin \theta \dot{\theta}, \\
I_{08} &= -s \dot{t}^2 + s a^2 \dot{r}^2 + s a^2 r^2 \dot{\theta}^2 + s a^2 r^2 \sin^2 \theta \dot{\phi}^2 + t \dot{t} - r a^2 \dot{r}, \\
I_{09} &= -\frac{s^2 \dot{t}^2}{2} + \frac{s^2 a^2 \dot{r}^2}{2} + \frac{s^2 a^2 r^2 \dot{\theta}^2}{2} + \frac{s^2 a^2 r^2 \sin^2 \theta \dot{\phi}^2}{2} + s t \dot{t} - s r a^2 \dot{r} - \frac{1}{2}(t^2 - r^2), \\
I_{10} &= a^2 s \sin \theta \sin \phi \dot{r} + a^2 r s \sin \phi \cos \theta \dot{\theta} + s r a^2 \cos \phi \sin \theta \dot{\phi} - r \sin \theta \sin \phi, \\
I_{11} &= a^2 s \sin \theta \cos \phi \dot{r} + a^2 r s \cos \phi \cos \theta \dot{\theta} - s r a^2 \sin \phi \sin \theta \dot{\phi} - r \sin \theta \cos \phi, \\
I_{12} &= 2r \sin \theta \sin \phi \dot{t} - 2t a^2 \sin \theta \sin \phi \dot{r} - 2t r a^2 \cos \theta \sin \phi \dot{\theta} - 2a^2 r t \cos \phi \sin \theta \dot{\phi}, \\
I_{13} &= 2r \sin \theta \cos \phi \dot{t} - 2t a^2 \sin \theta \cos \phi \dot{r} - 2t r a^2 \cos \theta \cos \phi \dot{\theta} - 2a^2 r t \sin \phi \sin \theta \dot{\phi}.
\end{aligned}$$

### 3.2 For open universe ( $k = 1$ )

Using curvature  $k = 1$  in Eq. (8) the Lagrangian becomes

$$L = \dot{t}^2 - \left( \frac{a^2 \dot{r}^2}{1 - r^2} \right) - a^2 r^2 \dot{\theta}^2 - a^2 r^2 \dot{\phi}^2 \sin^2 \theta. \quad (23)$$

Using Eq. (23) in Eq. (5) and after some tedious calculation we get an over-determined system of linear PDEs.

#### 3.2.1 Minimal set of Noether operators

For arbitrary choice of  $a(t)$  one can obtain minimal number of Noether symmetries i.e.  $\mathbf{X}_{01} - \mathbf{X}_{02}$ .

#### 3.2.2 Three Noether operators

Some different forms of  $a(t)$  which posses three Noether operators are listed in Table: 1 and the set of Noether symmetries consist of  $\mathbf{X}_{01} - \mathbf{X}_{02}$  and

$$X_{03} = (-\sqrt{r-1}\sqrt{r+1} \cos \theta) \frac{\partial}{\partial r} + \left( \frac{\sqrt{r-1}\sqrt{r+1} \sin \theta}{r} \right) \frac{\partial}{\partial \theta}, \quad (24)$$

The corresponding integral of motion for  $X_{03}$  is:

$$I_{03} = \frac{2a^2 \dot{r}}{1 - r^2} (\sqrt{r-1}\sqrt{r+1} \cos \theta) - 2a^2 r \dot{\theta} (\sqrt{r-1}\sqrt{r+1} \sin \theta).$$

### 3.2.3 Five Noether operators

In this case, we will report some of the forms of cosmic scale factors which posses five Noether operators.

(1):  $a(t)=\alpha t + \beta$

where  $\alpha$  and  $\beta$  are arbitrary constant. For this case, we have  $\mathbf{X}_{01} - \mathbf{X}_{02}$ , (19), (20) and (24) with gauge function:

$$A = \frac{1}{2\alpha} \left[ (c_1 t^2 + 2c_4)\alpha + 2c_1 \beta t \right]. \quad (25)$$

### 3.2.4 Maximal set of Noether operators

One can obtain the following set of maximal Noether operators for  $a(t) = \text{constant}$  i.e.

$\mathbf{X}_{01} - \mathbf{X}_{02}$ , (22) and (24) where gauge function is  $A = c_2 t + c_3$ .

## 3.3 For closed universe ( $k = -1$ )

Using curvature  $k = -1$  in Eq. (8) the Lagrangian becomes

$$L = \dot{t}^2 - \left( \frac{a^2 \dot{r}^2}{1 + r^2} \right) - a^2 r^2 \dot{\theta}^2 - a^2 r^2 \dot{\phi}^2 \sin^2 \theta. \quad (26)$$

Using Eq. (26) in Eq. (5) and after some tedious calculation we get an over-determined system of linear PDEs.

### 3.3.1 Minimal set of Noether operators

If  $a(t)$  is arbitrary then we get minimal number of Noether symmetries i.e.  $\mathbf{X}_{01} - \mathbf{X}_{02}$ .

### 3.3.2 Five Noether operators

Some different forms of  $a(t)$  which are listed in Table: 1 posses five Noether operators i.e.  $\mathbf{X}_{01} - \mathbf{X}_{02}$  and others

$$X_{03} = (-\sqrt{1+r^2} \cos \phi \sin \theta) \frac{\partial}{\partial r} - \left( \frac{\sqrt{1+r^2} \cos \theta \cos \phi}{r} \right) \frac{\partial}{\partial \theta} + \left( \frac{\sqrt{1+r^2} \sin \phi}{r \sin \theta} \right) \frac{\partial}{\partial \phi}, \quad (27)$$

$$X_{04} = (\sqrt{1+r^2} \sin \phi \sin \theta) \frac{\partial}{\partial r} + \left( \frac{\sqrt{1+r^2} \cos \theta \sin \phi}{r} \right) \frac{\partial}{\partial \theta} + \left( \frac{\sqrt{1+r^2} \cos \phi}{r \sin \theta} \right) \frac{\partial}{\partial \phi}, \quad (28)$$

$$X_{05} = (-\sqrt{1+r^2} \cos \theta) \frac{\partial}{\partial r} + \left( \frac{\sqrt{1+r^2} \sin \theta}{r} \right) \frac{\partial}{\partial \theta}. \quad (29)$$

The energy type first integral of motions associated with above generators are

$$I_{03} = \frac{2a^2 \dot{r}}{\sqrt{1+r^2}} \cos \phi \sin \theta + 2a^2 r \sqrt{1+r^2} \cos \theta \cos \phi \dot{\theta} - 2a^2 r \sqrt{1+r^2} \sin \theta \sin \phi \dot{\phi},$$

$$I_{04} = -\frac{2a^2 \dot{r}}{\sqrt{1+r^2}} \sin \phi \sin \theta - 2a^2 r \sqrt{1+r^2} \cos \theta \sin \phi \dot{\theta} - 2a^2 r \sqrt{1+r^2} \sin \theta \cos \phi \dot{\phi},$$

$$I_{05} = \frac{2a^2 \dot{r}}{\sqrt{1+r^2}} \cos \theta - 2a^2 r \sqrt{1+r^2} \sin \theta \dot{\theta}.$$

### 3.3.3 Maximal set of Noether operators

(1):  $a(t) = \alpha t + \beta$

where  $\alpha$  and  $\beta$  are arbitrary constant. For this the Noether operators are:

$\mathbf{X}_{01} - \mathbf{X}_{02}$ , (19), (20) and (27)-(29) with the gauge function:

$$A = \frac{1}{2\alpha} \left[ (c_1 t^2 + 2c_4) \alpha + 2c_1 \beta t \right]. \quad (30)$$

(2): If  $a(t)$  is constant then we have

$\mathbf{X}_{01} - \mathbf{X}_{02}$ , (22) and (27)-(29), where

$$A = c_2 t + c_3.$$

It should be noted that  $\mathbf{X}_{01} - \mathbf{X}_{02}$  not only independent from the choice of cosmic scale factor but also from the values of  $k$ .

## 4 Conclusion

In this paper, a classification of scale factor by considering different forms of FRW universe i.e. flat, open and closed was reported by using Noether approach. The Noether symmetries in the presence of gauge term was computed. We distributed our results with respect to different natures of universe. For the classification purpose, and an unusual Lagrangian for FRW metric was considered. With the help of considered Lagrangian and Noether equation, an over-determined system of PDEs was obtained. Then these systems were solved for different values of cosmic scale factor  $a(t)$ . Cases were reported with respect to different number of Noether operators. For all three natures of universe minimal and maximal set of Noether operators were computed. For each calculated Noether operator corresponding energy type first integral of motion was presented. It was reported that  $\mathbf{X}_{01} - \mathbf{X}_{02}$  not only independent from the choice of cosmic scale factor but also the natures of universe.

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